MAC 2311
Chapter 4
Spring 2020–– Kincade

Name: KEY
Date:

You MUST show your work to receive any credit. This part of the exam is worth 100 points. Each problem is worth 6 points unless otherwise specified.

Solve the problem.

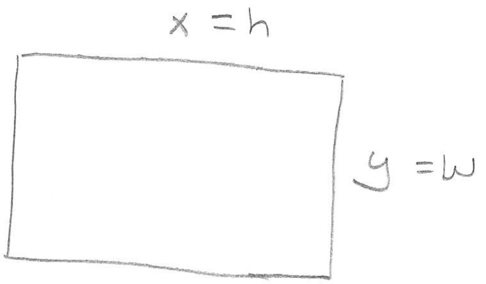
1) A private shipping company will accept a box for domestic shipment only if the sum of its length and girth (distance around) does not exceed 120 in. Suppose you want to mail a box with square sides so that its dimensions are h by h by w and it's girth is 2h + 2w. What dimensions will give the box its largest volume? 1) B

A) 20 in. × 20 in. × 40 in.

B) $\boxed{\frac{80}{3} \text{ in.} \times \frac{80}{3} \text{ in.} \times 20 \text{ in.}}$

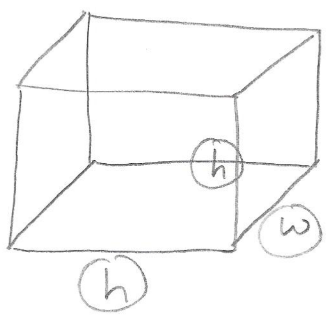
C) 40 in. × 20 in. × 40 in.

D) 20 in. × 20 in. × 100 in.



girth = perimeter

$\underbrace{2h + 2w + \underset{(h)}{h}}_{\text{sum of length + girth}} = 120$



$V = h \cdot h \cdot w = h^2 w$

$2h + 2w + h = 120$

$2w + 3h = 120$

$2w = 120 - 3h$

$w = 60 - \frac{3}{2}h$

$V = h \cdot h \cdot (60 - \frac{3}{2}h)$

$V = 60h^2 - \frac{3}{2}h^3$

$V' = 120h - \frac{9}{2}h^2$

$V' = 0 \quad h(120 - \frac{9}{2}h) = 0$

$h = 0 \quad 120 - \frac{9}{2}h = 0$

$120 = \frac{9}{2}h$

$240 = 9h$

$\frac{240}{9} = \frac{80}{3} = h$ inches

$w = 60 - \frac{3}{2}(\frac{80}{3})$

$w = 60 - 40 = 20 \text{ in}$

$\boxed{\frac{80}{3} \text{ in} \times \frac{80}{3} \text{ in} \times 20 \text{ in}}$

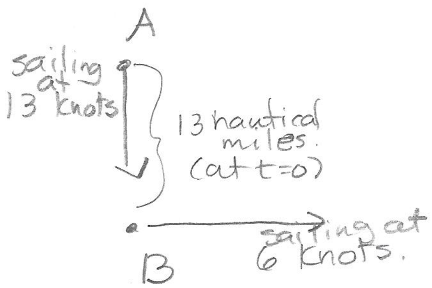
2) At noon, ship A was 13 nautical miles due north of ship B. Ship A was sailing south at 13 knots (nautical miles per hour; a nautical mile is 2000 yards) and continued to do so all day. Ship B was sailing east at 6 knots and continued to do so all day. The visibility was 5 nautical miles. Did the ships ever sight each other? (12 points) 2) C

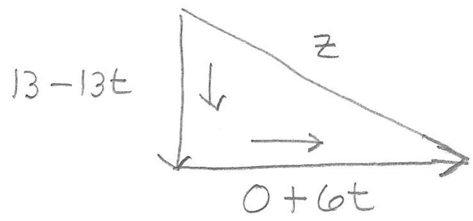
A) Yes. They were within 4 nautical miles of each other.

B) No. The closest they ever got to each other was 6.4 nautical miles.

C) No. The closest they ever got to each other was 5.4 nautical miles.

D) Yes. They were within 3 nautical miles of each other.





$$z^2 = (13-13t)^2 + (0+6t)^2$$

$$z^2 = 169 - 338t + 169t^2 + 36t^2$$

$$z^2 = 205t^2 - 338t + 169$$

$$z = (205t^2 - 338t + 169)^{1/2}$$

$$z' = \frac{1}{2}(205t^2 - 338t + 169)^{-1/2}(410t - 26)$$

$$z' = \frac{410t - 338}{2(205t^2 - 338t + 169)^{1/2}}$$

$\boxed{z' = 0}$

$$410t - 338 = 0$$

$$t = \frac{338}{410} \approx 0.82 \text{ hrs.}$$

$\boxed{z' \text{ undefined}}$

$$205t^2 - 338t + 169 = 0$$

$$t = \frac{338 \pm \sqrt{(-338)^2 - 4(205)(169)}}{2(205)}$$

not real (negative)

z'

− | +

0.82

min

$$z^2 = (13 - 13(0.82))^2 + (6(0.82))^2 = 29.68$$

$$z = \sqrt{29.68} \approx \boxed{5.44 \text{ nautical miles minimum}}$$

Use l'Hopital's Rule to evaluate the limit.

3) $\lim\limits_{x \to -\infty} \dfrac{16 + 2x - 19x^2}{3 - 8x - 7x^2}$ $\qquad \dfrac{\infty}{\infty}$ $\qquad$ 3) $\dfrac{19}{7}$

$$\overset{L}{=} \lim_{x \to \infty} \frac{2 - 38x}{-8 - 14x} \overset{L}{=} \lim_{x \to \infty} \frac{-38}{-14} = \frac{19}{7}$$

4) $\lim\limits_{x \to \infty} \left(\sqrt{x^2 + 4x} - x\right)$ $\qquad$ form $\infty - \infty$ $\qquad$ 4) 2

$$\lim_{x \to \infty} \frac{(\sqrt{x^2+4x} - x)(\sqrt{x^2+4x} + x)}{(\sqrt{x^2+4x} + x)}$$

form $\dfrac{\infty}{\infty}$

$$= \lim_{x \to \infty} \frac{x^2 + 4x - x^2}{\sqrt{x^2+4x} + x} = \lim_{x \to \infty} \frac{4x}{\sqrt{x^2+4x} + x}$$

$$\overset{L}{=} \lim_{x \to \infty} \frac{4}{\frac{1}{2}(x^2+4x)^{-1/2}(2x+4) + 1} = \lim_{x \to \infty} \frac{4}{\frac{2x+4}{2\sqrt{x^2+4x}} + 1} \left(\frac{2\sqrt{x^2+4x}}{2\sqrt{x^2+4x}}\right)$$

$$= \lim_{x \to \infty} \frac{4 \cdot 2\sqrt{x^2+4x}}{(2x+4) + 2\sqrt{x^2+4x}} \to \lim_{x \to \infty} \frac{8x}{4x} = 2$$

5) $\lim\limits_{x \to \infty} x \sin \dfrac{14}{x}$ $\qquad$ form $\infty \cdot 0$ $\qquad$ 5) 14

$$\lim_{x \to \infty} \frac{\sin \frac{14}{x}}{\frac{1}{x}} \qquad \frac{14x^{-1}}{x^{-1}}$$

$$\overset{L}{=} \lim_{x \to \infty} \frac{\cos\left(\frac{14}{x}\right)(-14x^{-2})}{-x^{-2}}$$

$$= \lim_{x \to \infty} \frac{\cos\left(\frac{14}{x}\right)\left(\frac{-14}{x^2}\right)}{\frac{-1}{x^2}}$$

$$= \lim_{x \to \infty} \frac{-x^2}{1} \cdot \cos\left(\frac{14}{x}\right)\left(\frac{-14}{x^2}\right)$$

$$= \lim_{x \to \infty} 14 \underbrace{\cos\left(\frac{14}{x}\right)}_{\to 1} = 14$$

Use l'Hopital's Rule to evaluate the limit.

6) $\lim\limits_{x \to 0} \dfrac{\sin 2x}{\tan 8x}$ $\quad \frac{0}{0}$ form $\qquad$ 6) $\frac{1}{4}$

$$\stackrel{L}{=} \lim_{x \to 0} \frac{2\cos 2x}{8\sec^2 2x} = \frac{2\cos(0)}{8\sec^2(0)} = \frac{2(1)}{8(1)} = \frac{1}{4}$$

Find the limit.

7) $\lim\limits_{x \to \infty} \left(1 + \dfrac{2}{x^2}\right)^x$ $\quad$ form 1^∞ $\qquad$ 7) 1

$$y = \lim_{x \to \infty} \left(1 + \frac{2}{x^2}\right)^x$$

$$\ln y = \lim_{x \to \infty} \ln\left(1 + \frac{2}{x^2}\right)^x$$

$$\ln y = \lim_{x \to \infty} \underbrace{x \ln\left(1 + \frac{2}{x^2}\right)}_{0 \cdot \infty}$$

$$\ln y = \lim_{x \to \infty} \frac{\ln(1 + 2x^{-2})}{x^{-1}}$$

$$\ln y \stackrel{L}{=} \lim_{x \to \infty} \frac{\frac{1}{1 + \frac{2}{x^2}} \cdot -4x^{-3}}{-x^{-2}}$$

$$\ln y = \lim_{x \to \infty} \left(\frac{x^2}{x^2}\right) \frac{1}{1 + \frac{2}{x^2}} \cdot \frac{-4}{x^3}$$

$$\ln y = \lim_{x \to \infty} \frac{x^2}{x^2 + 2} \cdot \frac{-4}{x^3}$$

$$\ln y = \lim_{x \to \infty} \frac{-4}{x(x^2+2)} \to 0$$

$$\ln y = 0$$

$$\log_e y = 0$$

$$y = e^0 = 1$$

Find the absolute extreme values of the function on the interval. (6 points)

8) $g(x) = -x^2 + 11x - 30, \; 6 \le x \le 5$ $\quad 5 \le x \le 6$ 8) B

A) absolute maximum is $\frac{241}{4}$ at $x = \frac{11}{2}$; absolute minimum is 0 at 5 and 0 at x = 6

B) absolute maximum is $\frac{1}{4}$ at $x = \frac{11}{2}$; absolute minimum is 0 at 5 and 0 at x = 6

C) absolute maximum is $\frac{1}{4}$ at $x = \frac{13}{2}$; absolute minimum is 0 at 5 and 0 at x = 6

D) absolute maximum is $\frac{5}{4}$ at $x = \frac{13}{2}$; absolute minimum is 0 at 5 and 0 at x = 6

test
$g'(5.4) = +$
$g'(5.6) = -$

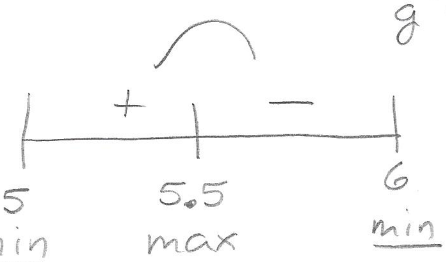
$g'(x) = -2x + 11$

$g'(x) = 0$ $\qquad$ $g'(x)$ undefined (none)

$-2x + 11 = 0$

$-2x = -11$

$x = \frac{11}{2}$



$g(5) = 0 \leftarrow$ tie for absolute min

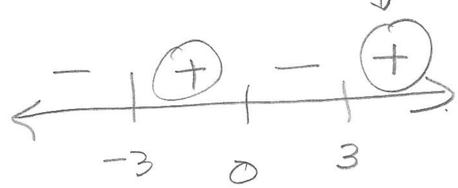
$g(5.5) = 0.25 \leftarrow$ absolute max

$g(6) = 0 \leftarrow$ tie for absolute min

Find the largest open interval where the function is changing as requested. (6 points)

9) Increasing $\quad y = (x^2 - 9)^2$ $\qquad$ 9) $\boxed{(3, \infty)}$

largest



$y = (x^2 - 9)^2$

$y' = 2(x^2 - 9)(2x)$

$y' = 4x(x+3)(x-3)$

critical numbers

$x = 0 \qquad x = -3 \qquad x = 3$

test

$y'(-4) = (-)(-)(-) = -$

$y'(-1) = (-)(+)(-) = +$

$y'(1) = (+)(+)(-) = -$

$y'(4) = (+)(+)(+) = +$

Determine where the given function is concave up and where it is concave down. **(6 points)**

10) $f(x) = 2x^3 + 9x^2 + 12x$

10) ____________

$f(x) = 2x^3 + 9x^2 + 12x$

$f'(x) = 6x^2 + 18x + 12$

$f''(x) = 12x + 18$

$f''(x) = 0$

$12x + 18 = 0$

$12x = -18$

$x = -\frac{18}{12}$

$x = -\frac{3}{2}$

f'': $-$ | $+$ at -1.5

$f''(-2) = -$

$f''(-1) = +$

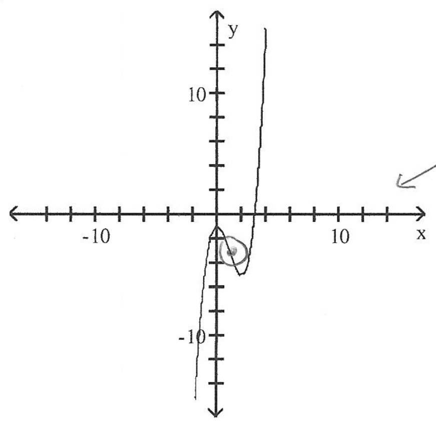
concave down $(-\infty, -\frac{3}{2})$

concave up $(-\frac{3}{2}, \infty)$

Use the graph of the function f(x) to locate the local extrema and identify the intervals where the function is concave up and concave down. (4 points)

11)

11) ____________



note scaled by 2's.

min $x = 2$

max $x = 0$

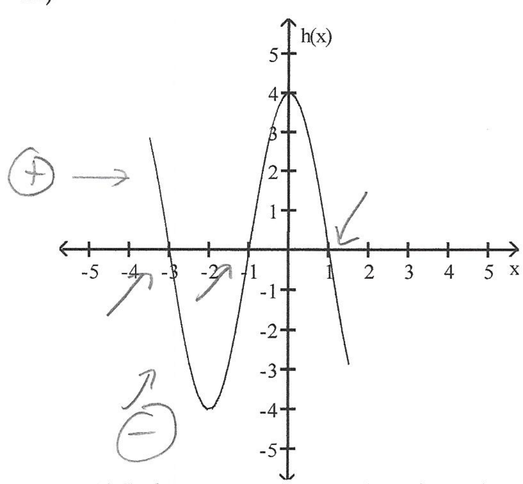
concave up $(1, \infty)$

concave down $(-\infty, 1)$

Suppose that the function with the given graph is not f(x), but f'(x). Find the locations of all extrema, and tell whether each extremum is a relative maximum or minimum. (4 bonus points) — Challenge Problem

12)

12) D



$h'(x)$

$+$ | $-$ | $+$ | $-$ at -3, -1, 1

max min max

A) Relative minima at −3 and 1; relative maximum at −1

B) Relative maximum at 0; relative minimum at −2

C) No relative extrema

D) Relative maxima at −3 and 1; relative minimum at −1

(13) Given $f(x) = \dfrac{x^3}{(x+1)^2}$ $f'(x) = \dfrac{x^2(x+3)}{(x+1)^3}$

and $f''(x) = \dfrac{6x}{(x+1)^4}$

$\boxed{\text{20 points total}}$

(a) Find all intercepts (4 points)
(b) Find all asymptotes (4 points)
(c) First derivative analysis (4 points)
(d) Second derivative analysis (4 points)
(e) Sketch a graph of the function. (4 points)

(13) $f(x) = \frac{x^3}{(x+1)^2}$ $f'(x) = \frac{x^2(x+3)}{(x+1)^3}$ $f''(x) = \frac{6x}{(x+1)^4}$

$\boxed{\text{(a) Intercepts (original function)}}$ $\boxed{f(x) = \frac{x^3}{(x+1)^2}}$

($y=0$) $x^3 = 0$ ($x=0$) $\frac{0}{(0+1)^2} = 0$

$x = 0$

$\boxed{(0,0)}$

$\boxed{\text{(b) Asymptotes (original function)}}$

$\boxed{VA}$ $(x+1)^2 = 0$ $\boxed{HA}$ $n > d \Rightarrow$ none

$x+1 = 0$

$\boxed{x = -1}$

$\boxed{\text{other}}$ (long division) $\boxed{y = x-2}$

$$
\begin{array}{r|l}
 & x - 2 \\
x^2+2x+1 & x^3 + 0x^2 + 0x + 0 \\
 & -(x^3 + 2x^2 + x) \\
 & -2x^2 - x + 0 \\
 & -(-2x^2 - 4x - 2) \\
 & 3x + 2
\end{array}
$$

(c) first derivative analysis

$$f'(x) = \frac{x^2(x+3)}{(x+1)^3}$$

$\boxed{f'(x) = 0}$ (num $= 0$)

$x^2(x+3) = 0$

$\boxed{x = 0}$ $\boxed{x = -3}$

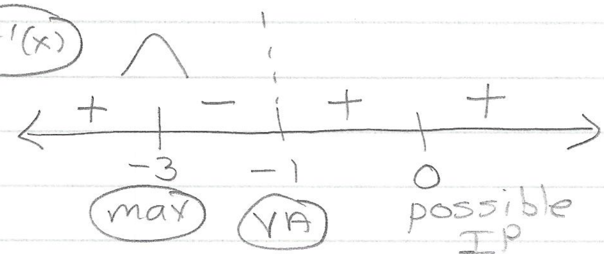
$\boxed{f'(x) \text{ undef}}$ (denom $= 0$)

$x + 1 = 0$

$\boxed{x = -1}$

VA

$f'(x)$



decreasing $(-3, -1)$

increasing $(-\infty, -3)$
$(-1, 0)$
$(0, \infty)$

$f'(-4) = (-)/(-) = +$

$f'(-2) = (+)/(-) = -$

$f'(-\tfrac{1}{2}) = (+)/(+) = +$

$f'(1) = (+)/(+) = +$

(d) second derivative analysis

$$f''(x) = \frac{6x}{(x+1)^4}$$

$\boxed{f''(x) = 0}$ (num)

$6x = 0$

$x = 0$

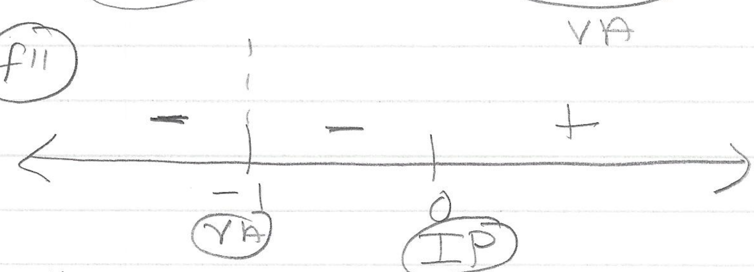
$\boxed{f''(x) \text{ undef}}$ (denom)

$x + 1 = 0$

$x = -1$

VA

f''



concave up $(0, \infty)$

concave down
$(-\infty, -1)$ & $(-1, \infty)$

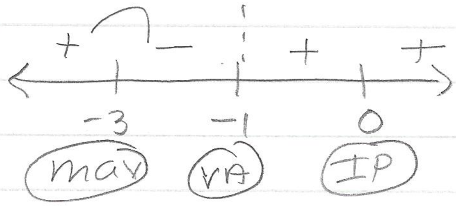
$f''(-2) = -$

$f''(-\tfrac{1}{2}) = -$

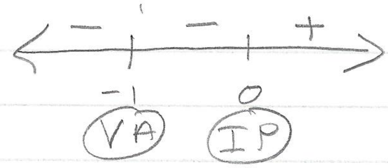
$f''(1) = +$

e The graph

f'



f''



points $f(x)$

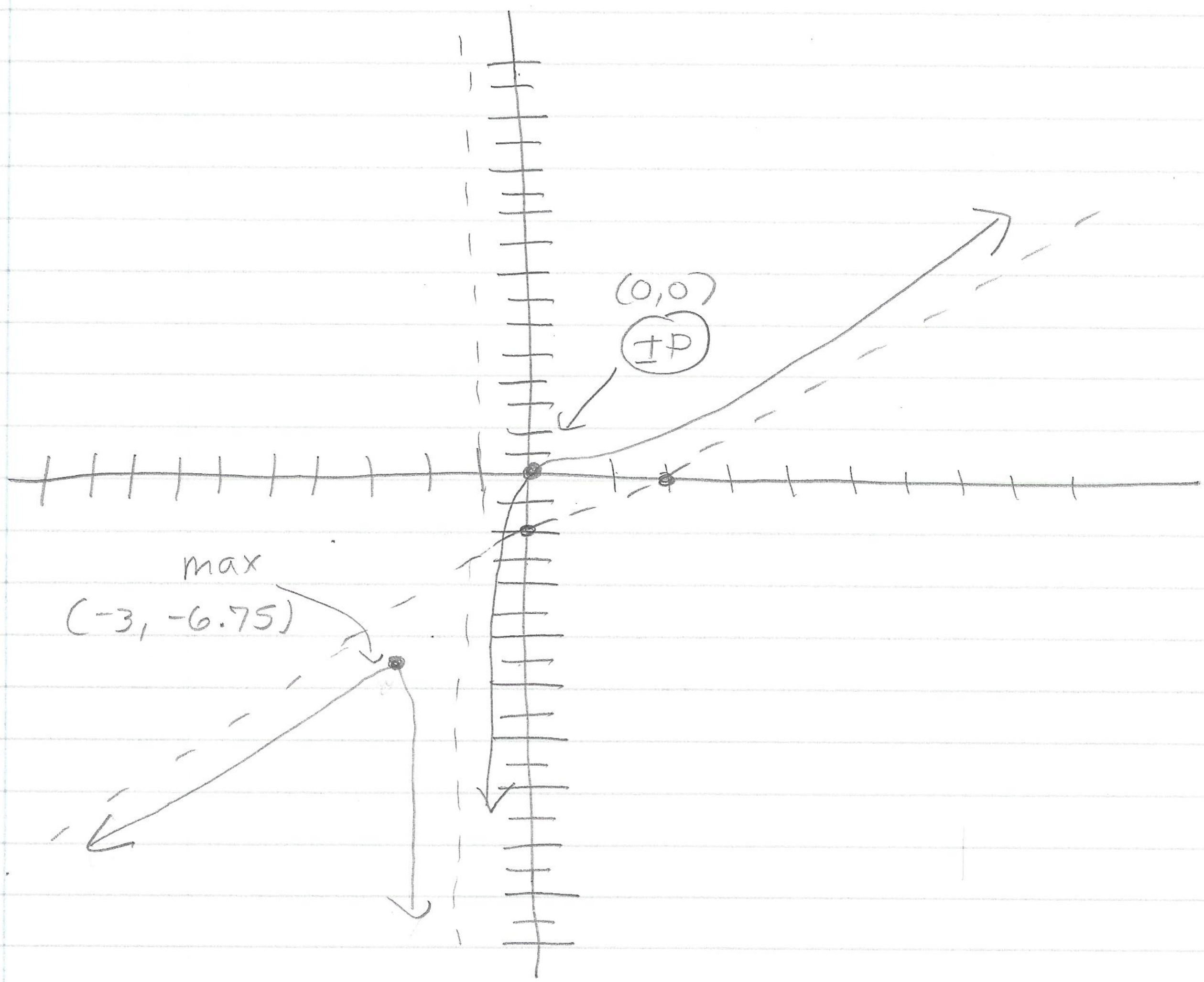
x	y $f(x)$	
−3	−6.75	max
0	0	IP & x-int

$\boxed{VA}$ $x = -1$

$$f(x) = \frac{x^3}{(x+1)^2}$$

x	y
0	−2
2	0

slant asymptote $y = x - 2$



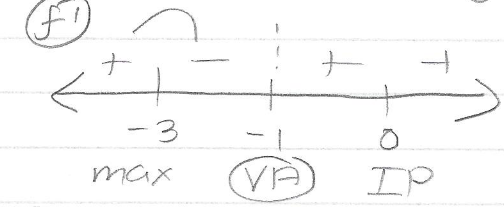
Summary Results for #13

(a) intercepts (0,0)

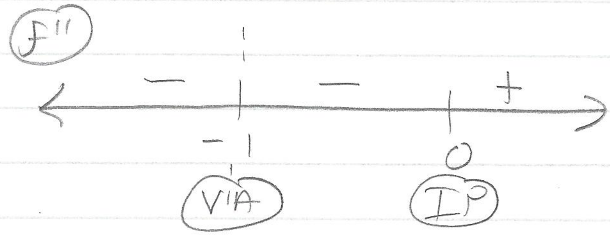
(b) asymptotes

	VA	HA	oblique
	$x=-1$	none	$y=x-2$

(c) max
$(-3, -6.75)$



(d) IP
$(0,0)$



(e) graph